Proceedings of the 2003 IEEE International Frequency Control Symposium and PDA Exhibition
Jointly with the 17th European Frequency and Time Forum

Time Scales Demystified

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***Abstract*-This paper explains the concept of a time-scale algorithm. It goes on to show that the non-observability of the time error of a clock results in ambiguity that produces method dependent solutions. Finally, we present a technique to remove the ambiguity and use simulation to show that the resulting algorithms span the performance range of algorithms reported in the literature.**
***Keywords*-Time scale, algorithm, simulator, atomic clock.**

I. INTRODUCTION

A time-scale algorithm is a mathematical method of estimating the time error of each clock in an ensemble of clocks. The time displayed by the clock is corrected by the estimated error before it is used because the corrected time is more uniform than the displayed time. Time-scale algorithms have been used this way for approximately 40 years [1]. During that time, the authors have frequently observed unexplained behavior [2,3]. This paper demonstrates that such unexplained behavior results from a failure to fully specify the problem rather than idiosyncrasies of the algorithms.

A clock has highly integrated noise because the clock itself behaves as a multiple integrator: The time is the integral of the frequency, which itself is an integral of the aging. Therefore, before considering how to use an ensemble of clocks, it makes sense to consider how to use an ensemble of *N* identical voltage references. There are no other voltage references in the lab and we believe that the "average" of the *N* devices fluctuates less than a single device. Unlike clocks, a simple model for the voltage references is that they are constant over time. We would like to average them, but all we have is a noiseless potentiometer, which allows us to measure each reference with respect to one reference in the ensemble. Thus we know V_2-V_1, V_3-V_1,…,V_N-V_1. We also know that the noise on the voltage references is random and uncorrelated. Unfortunately, this is not enough information to solve the problem.

If the ensemble were infinitely large, then the sum of all the voltage difference measurements would tend to zero. Although we realize that it is not true, we assume that the sum of the voltage errors of the *N* references is zero and therefore that the sum of the *N-1* voltage difference measurements is zero each time we make the measurements. This assumption removes the ambiguity, since we now have *N-1* measurements and one assumption or *N* relations to use to determine the corrections for the *N* voltages. We conclude that the nominal reference voltage is just the mean of the individual reference voltages

$$\begin{aligned} V_i &= V_0 + \Delta V_i \\ V_0 &= \frac{1}{N}\sum_{i=1}^{N} V_i \\ \Delta V_i &= \sum_{j=1}^{N}\left(V_i - V_j\right) \end{aligned} \tag{1.1}$$

This result, although elementary, points out several important features of the problem. Since only difference measurements are available, the correction to be applied to an individual ensemble member is not an observable quantity. It may be calculated based on the assumption that the individual corrections sum to zero. The calculated corrections, which are estimates of the noise on each individual voltage reference, are not correct and can't be expected to have the correct statistics since the estimates are insensitive to any common mode component of the noise. Nevertheless, we see that the value of each corrected voltage reference is equal to the average of all of them. In the presence of white noise, the average is an unbiased and optimum (minimum squared-error) estimator of the true mean of an infinite ensemble of identical voltage references.

II. THE TIME-SCALE PROBLEM

Several features of the time-scale problem differ from the simple problem of averaging voltage references described above. The members of the ensemble may have significantly different designs and noise properties, making a simple unweighted average inappropriate. Additionally, the phase noise of each clock results not only from direct phase perturbations, but also from perturbations of the frequency and frequency aging. Thus, there are additional ambiguities that that relate to how the frequency and frequency aging noises divide between the clocks.

III. THE CLOCK MODEL

A simple, very realistic model for all precision clocks is the perfect integrator. Hydrogen maser, cesium, and rubidium atomic clocks can all be modeled with no more than three states: the phase, $x(t)$, the frequency, $y(t)$, and the frequency aging, $\omega(t)$. The equations of motion that relate the states are

$$\begin{aligned} x(t_{k+1}) &= x(t_k) + \delta y(t_k) + \frac{\delta^2}{2}\omega(t_k) + \varepsilon(t_k) \\ y(t_{k+1}) &= y(t_k) + \delta\omega(t_k) + \eta(t_k) \\ \omega(t_{k+1}) &= \omega(t_k) + \alpha(t_k) \end{aligned} \tag{3.1}$$

Work supported by the Office of Naval Research, Contract No. N00014-02-C-0455

where t_k and t_{k+1} are successive sample times related by

$$t_{k+1} = t_k + \delta\,. \tag{3.2}$$

According to this model, the phase of the clock is the integral of its frequency, which in turn is an integral of the frequency aging. In addition, each state of the clock evolves from one time to the next by absorbing or integrating a random shock. The clock has random-walk phase noise (white frequency noise), random-walk frequency noise, and random-walk frequency aging noise. This model is sufficiently rich to describe all precision atomic clocks – active and passive H masers as well as Cs and Rb clocks.

The clocks are measured at each sample time, t_k. We denote these measurements

$$z_{ij}(t_k) = x_i(t_k) - x_j(t_k) + v(t_k) \tag{3.3}$$

where z_{ij} is the phase difference measurement between the i^{th} and j^{th} clocks and v is the noise perturbing the measurement.

IV. The Clock-Difference Solution

The time-scale problem is to estimate the states of each of the clocks. Just as in the case of the voltage reference example, this problem is ambiguous as described. The process of removing the ambiguity begins analogously to the previous example by assuming that the noises on the phase states of the individual clocks sum to zero. Thus,

$$\sum_{i=1}^{N} a_i\left(t_k\right)\hat{\varepsilon}_i\left(t_k\right) = 0\,, \tag{4.1}$$

where $\hat{\varepsilon}_i\left(t_k\right)$ are the estimated phase shocks and the a_i are weights that are introduced to account for the different noise levels of the clocks. For example, the random-walk phase-noise of a high-performance Cs clock is 10-dB lower than the random-walk phase-noise of a standard-performance Cs clock.

At this point, the solution to the voltage reference problem was immediate because the voltage references were assumed constant in time. However, the time-scale problem is more difficult because it is necessary to estimate how much of a measured phase change is due to the noise on the phase state and how much results from the integrated noises on the frequency and frequency aging states. Fortunately, this is a well-understood problem and there are numerous solutions using techniques such as Kalman filters and ARIMA models. These two approaches are examples of minimum squared-error estimation methods. All such methods are equally good since they produce estimates that have the minimum squared-error when a large number of solutions are considered. Of course, any approach may produce a better solution than the others for a single test case.

The clock-difference problem is set up by differencing equation (3.1) for two clocks i and j.

$$\begin{aligned} x_{ij}\left(t_{k+1}\right) &= x_{ij}\left(t_k\right) + \delta y_{ij}\left(t_k\right) + \frac{\delta^2}{2}\omega_{ij}\left(t_k\right) + \varepsilon_{ij}\left(t_k\right) \\ y_{ij}\left(t_{k+1}\right) &= y_{ij}\left(t_k\right) + \delta\omega_{ij}\left(t_k\right) + \eta_{ij}\left(t_k\right) \\ \omega_{ij}\left(t_{k+1}\right) &= \omega_{ij}\left(t_k\right) + \alpha_{ij}\left(t_k\right) \end{aligned} \tag{4.2}$$

The goal is to estimate the clock difference states given the model of equations (4.2) and the measurements of equation (3.3). This is a well-known problem and has been solved by several authors using different techniques. Barnes obtained the steady-state solution using the ARIMA approach (described in [4]) and Tryon and Jones [5] obtained a dynamic solution including the effects of startup and variable δ.

These statistical techniques estimate how much of the observed (measured) change in phase difference results from random phase shocks versus the quantity originating in frequency or frequency-aging shocks. It is easy to see how this happens in certain simple cases. Consider the example of a Cs-like clock which may be modeled as having only random walk phase and random-walk frequency noise over sample times from 100 seconds to at least 10 days or longer. Figure 1 shows the two-sample or Allan deviation of the clock. The $1/\sqrt{\tau}$ slope at times shorter than 10^6 seconds indicates that the phase deviations are almost entirely due to the random-walk phase noise. Conversely, the $\sqrt{\tau}$ slope at times longer than 10^7 seconds indicates that the phase deviations are almost entirely due to the random-walk frequency noise. At 3×10^6 seconds, the two noise processes make equal contributions to the observed deviations. However, this is only true in a statistical sense. Any individual phase change over the 3×10^6 second interval may be entirely due to a phase shock, entirely due to a frequency shock, or any combination of the two. The methods developed by Barnes, Tryon, and Jones make it possible to calculate estimates of the three noise processes: These estimates are designated: $\hat{\varepsilon}_{ij}\left(t_k\right)$, $\hat{\eta}_{ij}\left(t_k\right)$, and $\hat{\alpha}_{ij}\left(t_k\right)$.

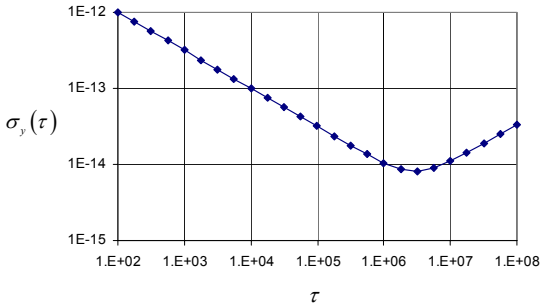

Figure 1: Allan deviation of a cesium clock

V. KALMAN-FILTER SOLUTION

Let us begin by defining a state vector that describes the time difference states

$$\underset{\sim}{x}_{ij}\left(t_k\right)=\begin{bmatrix} x_{ij}\left(t_k\right) \\ y_{ij}\left(t_k\right) \\ \omega_{ij}\left(t_k\right) \end{bmatrix}, \tag{5.1}$$

the noise vector that causes the states to evolve from one time to the next

$$\underset{\sim}{s}_{ij}\left(t_k\right)=\begin{bmatrix} \varepsilon_{ij}\left(t_k\right) \\ \eta_{ij}\left(t_k\right) \\ \alpha_{ij}\left(t_k\right) \end{bmatrix}, \tag{5.2}$$

and the state transformation matrix that embodies the equations of motion

$$\Phi=\begin{bmatrix} 1 & \delta & {}^{\delta^2}\!/_2 \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.3}$$

Equation (4.2) can now be written in the new notation:

$$\underset{\sim}{x}_{ij}\left(t_{k+1}\right)=\Phi\underset{\sim}{x}_{ij}\left(t_k\right)+\underset{\sim}{s}_{ij}\left(t_k\right). \tag{5.4}$$

The goal is to calculate a minimum squared-error estimate of the state vector assuming that the time difference is measured at each sample time. In order to use the Kalman filter approach it is necessary to define and calculate two covariance matrices. The plant covariance matrix is defined as

$$Q_{ij}=\left\langle \underset{\sim}{s}_{ij}\underset{\sim}{s}_{ij}^{T}\right\rangle, \tag{5.5}$$

where the angular brackets signify expectation value. The error covariance matrix is the covariance of the difference between the state vector and the estimate of the state vector. Thus,

$$P_{ij}\left(t_k\right)=\left\langle \left[\underset{\sim}{x}_{ij}\left(t_k\right)-\hat{\underset{\sim}{x}}_{ij}\left(t_k\right)\right]\left[\underset{\sim}{x}_{ij}\left(t_k\right)-\hat{\underset{\sim}{x}}_{ij}\left(t_k\right)\right]^{T}\right\rangle. \tag{5.6}$$

The Kalman approach uses the error covariance matrix just prior to a measurement and just after a measurement. These are denoted $P\left(t_k^-\right)$ and $P\left(t_k^+\right)$ respectively. Finally, we formally describe the measurements:

$$\underset{\sim}{z}_{ij}\left(t_k\right)=H_{ij}\underset{\sim}{x}_{ij}\left(t_k\right)+R_{ij}, \tag{5.7}$$

where H is a scalar equal to 1 and R is a scalar equal to the standard deviation of the measurements. Finally, we write the traditional form of the Kalman filter recursion:

$$\begin{aligned} P_{ij}\left(t_{k+1}^-\right)&=\Phi P_{ij}\left(t_k^+\right)\Phi^T+Q_{ij} \\ K_{ij}\left(t_{k+1}\right)&=P_{ij}\left(t_{k+1}^-\right){H_{ij}}^T\left[H_{ij}P_{ij}\left(t_{k+1}^-\right){H_{ij}}^T+R_{ij}\right]^{-1} \\ P_{ij}\left(t_{k+1}^+\right)&=\left[I-K_{ij}\left(t_{k+1}\right)H_{ij}\right]P_{ij}\left(t_{k+1}^-\right)\left[I-K_{ij}\left(t_{k+1}\right)H_{ij}\right]^T \\ &\quad +K_{ij}\left(t_{k+1}\right)R_{ij}K_{ij}^T\left(t_{k+1}\right) \\ \hat{\underset{\sim}{x}}_{ij}\left(t_{k+1}\right)&=\Phi\hat{\underset{\sim}{x}}_{ij}\left(t_k\right)+K_{ij}\left(t_{k+1}\right)\left[\underset{\sim}{z}_{ij}\left(t_{k+1}\right)-H\Phi\hat{\underset{\sim}{x}}_{ij}\left(t_k\right)\right] \end{aligned} \tag{5.8}$$

K_{ij} is the Kalman gain. It is a 3-element column vector:

$$K_{ij}=\begin{bmatrix} k_{ij}^x \\ k_{ij}^y \\ k_{ij}^\omega \end{bmatrix}. \tag{5.9}$$

The last line of equation (5.8) is the state update. Combine it with equation (5.4) to estimate the vector of random shocks:

$$\begin{aligned} \hat{\underset{\sim}{s}}_{ij}\left(t_k\right)&=\hat{\underset{\sim}{x}}_{ij}\left(t_{k+1}\right)-\Phi\hat{\underset{\sim}{x}}_{ij}\left(t_k\right) \\ &=K_{ij}\left(t_{k+1}\right)\left[\underset{\sim}{z}_{ij}\left(t_{k+1}\right)-H\Phi\hat{\underset{\sim}{x}}_{ij}\left(t_k\right)\right]. \end{aligned} \tag{5.10}$$

VI. THE BASIC TIME-SCALE EQUATION

When this process is completed for all clocks $j=1\ldots N$, $j\neq i$, there finally is enough information to solve for the individual clock phase shocks. This is obvious after examination of the N simultaneous equations for the N phase shocks.

$$\begin{aligned} &\sum_{i=1}^{N}a_i\left(t_k\right)\hat{\varepsilon}_i\left(t_k\right)=0 \\ &\hat{\varepsilon}_i\left(t_k\right)-\hat{\varepsilon}_j\left(t_k\right)=k_{ij}^x\left(t_{k+1}\right)\left[\underset{\sim}{z}_{ij}\left(t_{k+1}\right)-H\Phi\hat{\underset{\sim}{x}}_{ij}\left(t_k\right)\right],\ j\neq i \end{aligned} \tag{6.1}$$

and therefore

$$\hat{\varepsilon}_j\left(t_k\right)=\sum_{i=1}^{N}a_ik_{ij}^x\left(t_{k+1}\right)\left[H\Phi\hat{\underset{\sim}{x}}_{ij}\left(t_k\right)-\underset{\sim}{z}_{ij}\left(t_{k+1}\right)\right]. \tag{6.2}$$

At this point, it is instructive to try estimate the individual clock phases.

$$\begin{aligned} \hat{x}_i\left(t_{k+1}\right)&=\hat{x}_j\left(t_k\right)+\hat{y}_j\left(t_k\right)\delta+\hat{\omega}_j\left(t_k\right)\frac{\delta^2}{2}+\hat{\varepsilon}_j\left(t_k\right) \\ \hat{x}_j\left(t_{k+1}\right)&=\hat{x}_j\left(t_k\right)+\hat{y}_j\left(t_k\right)\delta+\hat{\omega}_j\left(t_k\right)\frac{\delta^2}{2} \\ &\quad +\sum_{i=1}^{N}a_ik_{ij}^x\left(t_{k+1}\right)\left[H\Phi\hat{\underset{\sim}{x}}_{ij}\left(t_k\right)-\underset{\sim}{z}_{ij}\left(t_{k+1}\right)\right] \end{aligned} \tag{6.3}$$

Equation (6.3) is often referred to as the basic time scale equation. However, it is clear that there is insufficient information to estimate the phase states of the individual

clocks since the prior frequency state, $\hat{y}_i(t_k)$, is unknown. Similarly, the prior frequency aging state, $\hat{\omega}_i(t_k)$, is also unknown. Previous time-scale algorithms have specified either explicitly or implicitly a method for estimating these states. This ambiguity has led to often-expressed confusion regarding the reason why a time-scale algorithm behaves in a certain way. The explanation for the confusion is that the time-scale algorithms have been ambiguously defined. The next section discusses how to eliminate the ambiguity and control the performance of the time-scale.

VII. The Supplemental Time-Scale Equations

The difference between the frequency shocks is known, but it is necessary to estimate the individual frequency shocks for each clock. The same requirement applies to the individual frequency aging random shocks for each clock. However, this can't be done because in each case there are only $N-1$ simultaneous equations for N unknowns. The ambiguity is eliminated by making the same assumption about the frequency and frequency-aging shocks that was made about the phase shocks in equation (4.1):

$$\begin{aligned} \sum_{i=1}^{N} b_i(t_k)\hat{\eta}_i(t_k) &= 0 \\ \sum_{i=1}^{N} c_i(t_k)\hat{\alpha}_i(t_k) &= 0 \end{aligned} \tag{7.1}$$

Now it is possible to solve for the frequency and frequency aging of each clock.

$$\begin{aligned} \hat{\eta}_j(t_k) &= \sum_{i=1}^{N} b_i k_{ij}^{y}(t_{k+1})\left[H\Phi\hat{x}_{ij}(t_k) - z_{ij}(t_{k+1})\right] \\ \hat{\alpha}_j(t_k) &= \sum_{i=1}^{N} c_i k_{ij}^{\omega}(t_{k+1})\left[H\Phi\hat{x}_{ij}(t_k) - z_{ij}(t_{k+1})\right] \end{aligned} \tag{7.2}$$

Solutions for the frequency and frequency aging are obtained by inserting the values for the random shocks from equation (7.2) into equation (5.4).

$$\begin{aligned} \hat{y}_j(t_{k+1}) &= \hat{y}_j(t_k) + \delta\hat{\omega}_j(t_k) \\ &\quad + \sum_{i=1}^{N} b_i k_{ij}^{y}(t_{k+1})\left[H\Phi\hat{x}_{ij}(t_k) - z_{ij}(t_{k+1})\right] \\ \hat{\omega}_j(t_{k+1}) &= \hat{\omega}_j(t_k) \\ &\quad + \sum_{i=1}^{N} c_i k_{ij}^{\omega}(t_{k+1})\left[H\Phi\hat{x}_{ij}(t_k) - z_{ij}(t_{k+1})\right] \end{aligned} \tag{7.3}$$

Equations (7.3) are the supplemental time scale equations. The need to make assumptions (7.1) that lead to these equations was first realized by this author a decade ago [6].

The procedure for calculating the time scale starts with initial values for the three states: x_0, y_0, and ω_0 Then, for each iteration, calculate the time, frequency, and frequency aging states. The ambiguity concerning how to do this has been removed and the algorithm that has been defined satisfies the limit theorem that the sum of the weighted random shocks approaches zero as the number of clocks gets large.

VIII. Results From Simulation

The algorithm of equations (6.3) and (7.3) has been tested using simulation. Two clocks were simulated directly from equation (3.1) using a Gaussian random number generator. The clocks are assumed to resemble Cs clocks and have only random-walk phase noise and random-walk frequency noise. The frequency aging state was assumed zero for all time. The measurements were simulated by differencing the clock phases. The problem was simplified by assuming the measurement noise is negligible compared to the clock noise. Thus, no measurement noise was added to the simulated measurements.

Figure 2 shows the performance of a the two-clock time scale when the phase and frequency weights are set according to the respective random walk phase noises of the two clocks. The units of time and the Allan deviation levels are arbitrary. The time scale tracks the clock with better short-term stability even at longer times when the other clock has better performance. The presence of the second clock improves the time-scale performance by 5% in the short term and 10% in the long term, but this is far from optimum performance.

Figure 3 shows the performance of a the two-clock time scale when the phase and frequency weights are set according to the respective random walk frequency noises of the two clocks. The time scale tracks the clock with better long-term stability even at shorter times when the other clock has better performance. The presence of the second clock improves the time-scale performance by 18% in the short term and 6% in the long term. This is also far from optimum performance.

Finally, Figure 4 shows the performance of a the two-clock time scale when the phase weights are set according to the respective random-walk phase noise and the frequency weights are set according to the random-walk frequency noise. The time scale is better than the best clock by 3% in short

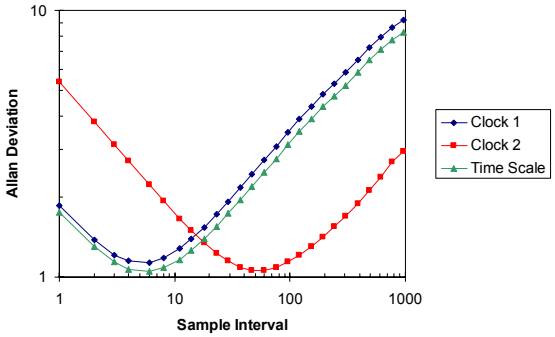

Figure 2: Two-clock time scale with the clocks weighted according to their random walk phase noise

term, but it is also better than the best clock in long term by 8%. Although "optimum" is subjective and depends on the needs of the user, this solution is much closer to "optimum" than either of the previous two solutions.

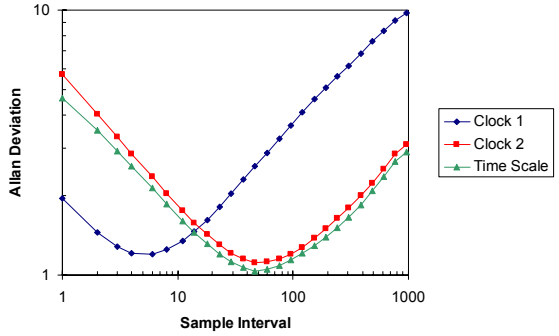

Figure 3: Two-clock time scale with the clocks weighted according to their random-walk frequency noise

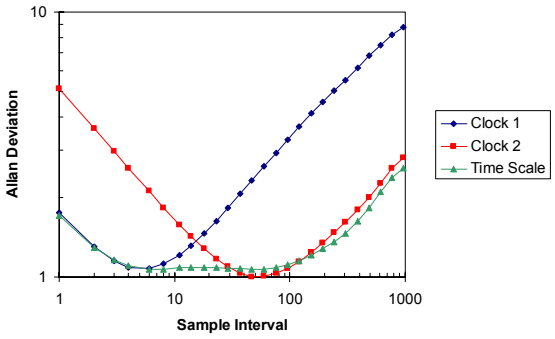

Figure 4: Two-clock time scale with "optimum" weights

IX. Conclusions

It has been shown that a time scale is nothing more than a method of estimating the individual noise inputs to a clock that is a member of an ensemble of several clocks. The problem is under defined because only clock time-difference measurements are available. The unexplained performance of previous algorithms resulted from the more-or-less serendipitous selection of one of the infinite number of solutions to this problem.

The approach presented in this paper removes the ambiguity by forcing the finite ensemble of clocks to have a property that is true in the limit of an infinite number of clocks: the weighted sum of the random noise inputs to the clocks is zero. The result is an algorithm whose performance may be optimized in each reason where a different noise type is dominant. Simulation studies have been used to show that the performance of this one algorithm may be adjusted over the full range of expected performance by varying the values of the weight parameters.

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